Mathematics

No.
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定義
$$\begin{cases} A: \text{Hermitian} \iff A^* = A \\ A: \text{Unitary} \iff A^*A = I \end{cases}$$

$$\begin{cases} A^* = (\overline{A})^T \\ e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \cdots \end{cases}$$

定理 1

(1) $\langle Ax, y \rangle = \langle x, A^*y \rangle$

(2) $A^{**} = A, \quad (\lambda A)^* = \overline{\lambda} A^*$

$(A+B)^* = A^* + B^*, \quad (AB)^* = B^*A^*$

(3) A 可逆 $\Rightarrow$ A^* 可逆, $(A^*)^{-1} = (A^{-1})^*$

$AB = I \Rightarrow B^*A^* = I$

$A = A^* \Rightarrow B = B^*$

定理 2

(1) $(e^A)^* = e^{A^*}$

(2) $AB = BA \Rightarrow e^A e^B = e^{A+B}$

証
$$e^A e^B = \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{j=0}^{\infty} \frac{B^j}{j!} = \sum_{n=0}^{\infty} \sum_{\substack{k+j=n \\ 0 \le k, j \le n}} \frac{1}{k!\,j!} A^k B^j = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} A^k B^{n-k} \right)$$

定理 3

$Ax = \lambda x \Rightarrow$ (1) $A^{-1}x = \lambda^{-1}x$

(2) $e^A x = e^\lambda x \quad \left(\sum_{k=0}^{n} \frac{A^k}{k!} x = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x = e^\lambda x \right)$

定理 4

A : Hermitian $\Rightarrow$ (0) A^{-1} : Hermitian

(1) $\forall$ eigen values $\in \mathbb{R}$, $\begin{cases} Ax = \lambda x \\ Ay = \mu y \end{cases}$ $\lambda \neq \mu \Rightarrow x \perp y$

(2) $\exists$ orthonormal eigen basis $\{u_1, \dots, u_n\}$

$$A = [u_1 \cdots u_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix} = \lambda_1 u_1 u_1^* + \cdots + \lambda_n u_n u_n^*$$
$$(U \Lambda U^*) \qquad = \lambda_1 |u_1\rangle\langle u_1| + \cdots + \lambda_n |u_n\rangle\langle u_n|$$

(3) A^{-1} : Hermitian

$$A^{-1} = [u_1 \cdots u_n] \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{bmatrix} \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix} = \lambda_1^{-1} u_1 u_1^* + \cdots + \lambda_n u_n u_n^*$$

定理 5

A : Hermitian $\Rightarrow$ e^{iA} unitary $\quad (e^{-iA}, e^{iAt}, e^{-iAt})$

証 $(e^{iA})^* e^{iA} = e^{-iA} e^{iA} = e^0 = I$

Rotational Operators

$$\begin{cases} R_z(\theta) &= e^{\frac{i\theta S_z}{\hbar}} = e^{i\frac{\theta}{2}\sigma_z} \\ R_x(\theta) &= e^{i\frac{\theta}{2}\sigma_x} \\ R_y(\theta) &= e^{i\frac{\theta}{2}\sigma_y} \end{cases}$$

$$e^{i\frac{\theta}{2}\sigma_j} = \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!}\sigma_j^n$$

$$\begin{cases} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

$$\begin{cases} R_z(\theta) = e^{i\frac{\theta}{2}\sigma_z} = \begin{pmatrix} \sum\limits_{n=0}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & 0 \\ 0 & \sum\limits_{n=0}^{\infty} \frac{\left(\frac{-i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \\ R_y(\theta) = \begin{pmatrix} \sum\limits_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & -i \sum\limits_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \\ i \sum\limits_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum\limits_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ R_x(\theta) = \begin{pmatrix} \sum\limits_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum\limits_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \\ \sum\limits_{n=1,3,5\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} & \sum\limits_{n=0,2,4\ldots}^{\infty} \frac{\left(\frac{i\theta}{2}\right)^n}{n!} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{cases}$$

$$\begin{cases} \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum\limits_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum\limits_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \end{cases}$$

HHL Algorithm (Harrow, Hassidim, Lloyd, '09)

- Given $\begin{cases} A : \text{Hermitian} \\ \mathbb{b} \end{cases}$, solve $A\mathbb{x} = \mathbb{b}$ $\quad (A = A^*)$

(1) $A \neq A^*$, solve $\underbrace{\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}}_{\text{Hermitian}} \begin{bmatrix} 0 \\ \mathbb{x} \end{bmatrix} = \begin{bmatrix} \mathbb{b} \\ 0 \end{bmatrix}$

$\exists \mathcal{B} = \{u_1, \dots, u_n\}$ orthonormal eigen basis

(2) A : Hermitian $\Rightarrow$ A^{-1} : Hermitian

$A\mathbb{x} = \lambda \mathbb{x}$ $\qquad A^{-1}\mathbb{x} = \frac{1}{\lambda}\mathbb{x}$

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}_{\mathcal{B}} \qquad A^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix}_{\mathcal{B}}$$

$$= \lambda_1 u_1 u_1^* + \cdots + \lambda_n u_n u_n^* \qquad = \frac{1}{\lambda_1} u_1 u_1^* + \cdots + \frac{1}{\lambda_n} u_n u_n^*$$

(3) $\mathbb{b} = \beta_1 u_1 + \cdots + \beta_n u_n$

$A\mathbb{x} = \mathbb{b} \Rightarrow \mathbb{x} = A^{-1}\mathbb{b}$

$$= \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$= \frac{\beta_1}{\lambda_1} u_1 + \cdots + \frac{\beta_n}{\lambda_n} u_n$$

$$= \sum_{j=1}^{n} \frac{\beta_j}{\lambda_j} |u_j\rangle$$

(4) $\begin{cases} A \\ A^{-1} \end{cases}$ not unitary, but $\begin{cases} e^{iA}, & e^{iAt} \\ e^{-iA}, & e^{-iAt} \end{cases}$ unitary

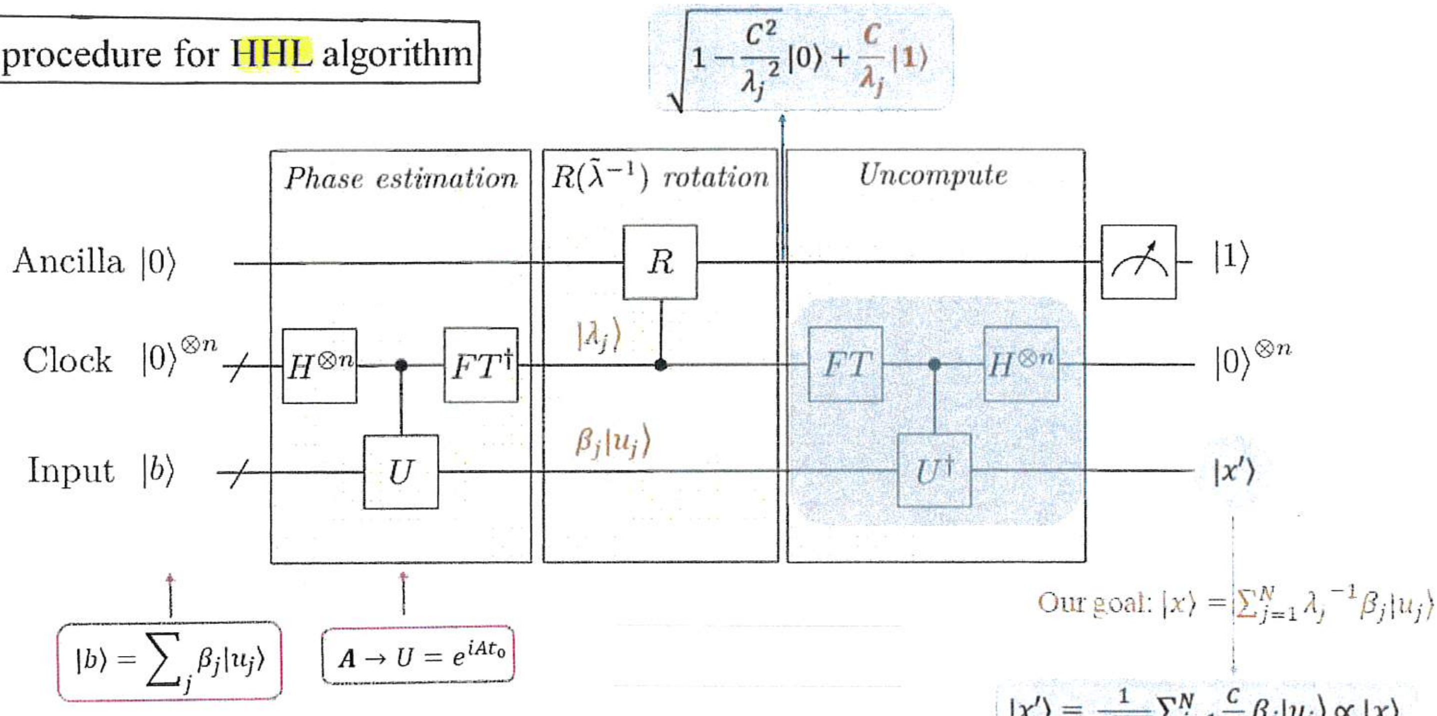
(Hamiltonian simulation)

(5) $\sin\frac{\theta}{2} = \frac{c}{\lambda}$ (Controlled rotation)

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} : |0\rangle \longrightarrow \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

$$= \sqrt{1 - \frac{c^2}{\lambda^2}}\,|0\rangle + \frac{c}{\lambda}|1\rangle$$

The procedure for HHL algorithm



$e^{i\lambda} = e^{2\pi i \frac{\lambda}{2\pi}}$

Algorithm 4 HHL algorithm

Input:

- The state $|b\rangle = \sum_j \beta_j |u_j\rangle$
- The ability to perform controlled operations with unitaries of the form e^{iAt}

Output:

- The quantum state $|x\rangle$ such that $A\vec{x} = \vec{b}$.

Procedure:

Step 1. Perform quantum phase estimation using the unitary transformation e^{iA}. This maps the eigenvalues λ_j into the register in the binary form to transform the system,

$$|0\rangle_a |0\rangle_r |b\rangle_m \to \sum_{j=1}^{N} \beta_j |0\rangle_a |\lambda_j\rangle_r |u_j\rangle_m .$$

Step 2. Rotate the ancilla qubit $|0\rangle_a$ to $\sqrt{1-\frac{C^2}{\lambda_j^2}}|0\rangle_a + \frac{C}{\lambda_j}|1\rangle_a$ for each λ_j. This is performed through controlled rotation on the $|0\rangle_a$ ancilla qubit. The system will evolve to

$$\sum_{j=1}^{N} \beta_j \left(\sqrt{1-\frac{C^2}{\lambda_j^2}}\,|0\rangle_a + \frac{C}{\lambda_j}|1\rangle_a\right)|\lambda_j\rangle_r |u_j\rangle_m .$$

Step 3. Perform the reverse of Step 1. This will lead the system to

$$\sum_{j=1}^{N} \beta_j \left(\sqrt{1-\frac{C^2}{\lambda_j^2}}\,|0\rangle_a + \frac{C}{\lambda_j}|1\rangle_a\right)|0\rangle_r |u_j\rangle_m .$$

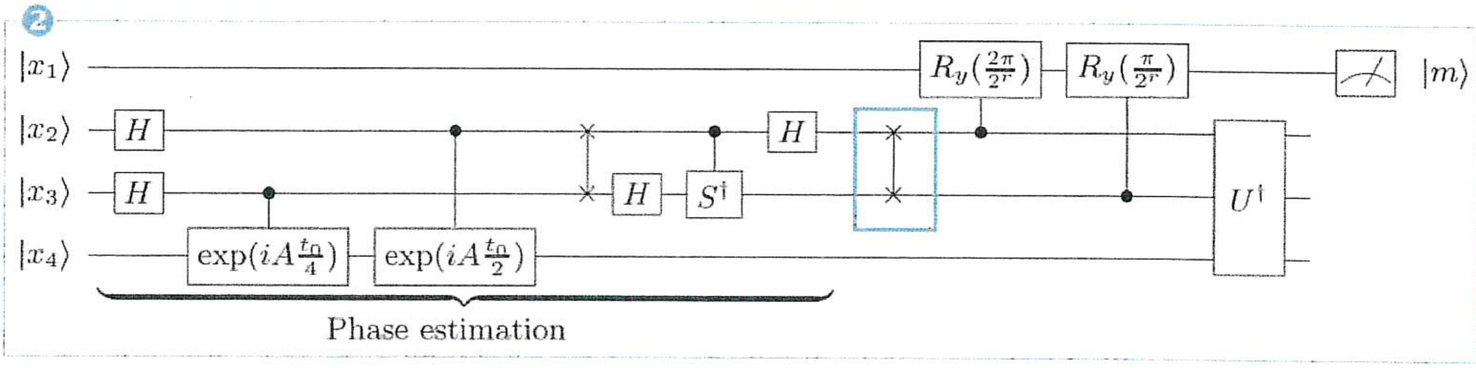
Step 4. Measuring the ancilla qubit will give ,

(amplitude amplification)

$$|x\rangle \approx \sum_{j=1}^{N} C\left(\frac{\beta_j}{\lambda_j}\right)|u_j\rangle ,$$

if the measurement outcome is $|1\rangle$

举例子



$A = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}; \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\lambda_1 = 1 : |x_2x_3\rangle = |01\rangle, \quad |u_1\rangle$

$\lambda_2 = 2 : |x_2x_3\rangle = |10\rangle, \quad |u_2\rangle$

$|b\rangle = b_1|0\rangle + b_2|1\rangle$

$\quad = \beta_1|u_1\rangle + \beta_2|u_2\rangle$

PE: $\boxed{\beta_1|\lambda_1\rangle|u_1\rangle + \beta_2|\lambda_2\rangle|u_2\rangle}$

$|x_2x_3x_4\rangle = \beta_1|01\rangle|u_1\rangle + \beta_2|10\rangle|u_2\rangle$

Swap:

$2\lambda_1^{-1} = 2 : |x_2x_3\rangle = |10\rangle$

$2\lambda_2^{-1} = 1 : |x_2x_3\rangle = |01\rangle$

$|x_2x_3x_4\rangle = \beta_1|10\rangle|u_1\rangle + \beta_2|01\rangle|u_2\rangle$

$\boxed{\beta_1|2\lambda_1^{-1}\rangle|u_1\rangle + \beta_2|2\lambda_2^{-1}\rangle|u_2\rangle}$

CR&M:

$|x_4\rangle = \beta_1|u_1\rangle + \frac{1}{2}\beta_2|u_2\rangle$

$\boxed{\lambda_1^{-1}\beta_1|u_1\rangle + \lambda_2^{-1}\beta_2|u_2\rangle}$

$A^{-1}b$

$$\begin{cases} \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} & \lambda_1 = 1 \quad |u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} & \lambda_2 = 2 \quad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

Example : $A_{4\times4}$